# Math 579 Fall 2013 Exam 3 Solutions

1. How many solutions to $a + b + c + d = 16$ are there in odd, nonnegative, integers?

   We write $a = 2a' + 1, b = 2b' + 1, c = 2c' + 1, d = 2d' + 1$, for nonnegative integers a', b', c', d'. Our equation becomes $2a' + 2b' + 2c' + 2d' = 12$, or $a' + b' + c' + d' = 6$. This last has $\left(\binom{4}{6}\right) = 84$ solutions; hence so does our original question.

2. In how many different ways can we place two white, two black, and four red rooks on a standard 8×8 chessboard so that no two rooks attack each other?

   We first choose positions for the rooks, which we can do in 8! ways. We next distinguish the rooks, and place them in these positions, which we can again do in 8! ways. Lastly, we consider equivalences between the artificially distinguished rooks; each equivalence class has 2!2!4! elements. Hence there are $\frac{8!8!}{2!2!4!} = 16,934,400$ different placements.

3. We want to select four subsets A, B, C, D of $[n]$ so that $A \subseteq (B \cup C) \subseteq D$ and $B \cap C = \emptyset$. In how many different ways can we do this?
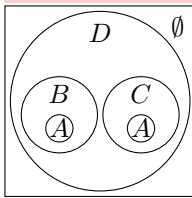

   The Venn diagram looks as left – two eggs frying in a pan, or perhaps a face with eyes but no nose or mouth. Set A has part (possibly none) inside of B, and part (possibly none) inside of C. Each element of $[n]$ must go into exactly one of the six regions displayed: none, D only, $D + B$ only, $D + C$ only, $D + B + A$ but not C, or $D + C + A$ but not B. Hence there are 6^n ways to choose these subsets.

4. How many four-digit positive integers contain exactly two different digits?

   Method 1: Two cases. First case: neither digit is zero. There are $\binom{9}{2} = 36$ ways to choose the two digits. Then, each position can be either the smaller or larger of the two digits, for $2^4 = 16$ possibilities, of which we must exclude two (all smaller digit, or all larger digit). Hence there are $36 \times 14 = 504$ in the first case. Second case: one of the digits is zero. There are $\binom{9}{1} = 9$ ways to choose the two digits. The leading position can't be zero. The remaining positions can be either digit, for $2^3 = 8$ possibilities, of which we must exclude the case of all nonzero-digit. Hence there are $9 \times 7 = 63$ in the second case. Adding, we get 567 altogether.

   Method 2: There are $\binom{10}{2} = 45$ ways to choose the two digits. Each position can be the larger or smaller of the two, but we exclude two of these, to get 14 as before. Hence there are $45 \times 14 = 630$ "numbers" of this type. However, we need to exclude those starting with zero. There are 9 ways to pick the other digit, and $2^3 - 1 = 7$ ways to fill the last three spots, so we need to subtract $9 \times 7 = 63$. Subtracting, we get $630 - 63 = 567$ for our answer.

5. How many 3×3 square matrices are there whose entries are 0 or 1 and which each row and column has an even sum?
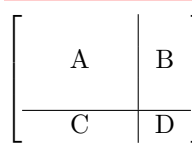
   We partition the matrix into a square block A, a 2×1 column vector B, a 1×2 row vector C, and a 1×1 entry D. We choose the entries of A freely, in $2^{(3-1)^2} = 16$ ways. The entries of B, C are determined to make the row/column sums correct. There is at most one possible choice for D; it remains to prove that one choice must work. A simple though tedious approach is just to list all 16 matrices.

   A better way, that works for $n \times n$ matrices too, is induction on the number of 1's in A. Base case: $A = 0$, then everything is 0 and $D = 0$ works. Otherwise, reset any single bit $a_{i,j}$ of A from 1 to 0; call this A'. By the inductive hypothesis there are corresponding B', C', D' that work with A'. We now get B, C, D by toggling (from 1 to 0 or vice versa) the ith entry of B', the jth entry of C', and D'. Each row and column had zero or two toggles, hence still has even sum.